Singular Kinetic SDEs

Zimo Hao
郝子墨
Bielefeld University
zhao@math.uni-bielefeld.de

19th Workshop on Markov Processes and Related Topics, Fuzhou

24 July, 2024

- Schauder estimates for nonlocal kinetic equations and applications.
  with **Mingyan Wu** and **Xicheng Zhang**
  J. Math. Pures. Appl. 140 (2020), 139–184.

- Singular kinetic equations and applications.
  with **Xicheng Zhang**, **Rongchan Zhu**, and **Xiangchan Zhu**
  Ann. Probab. 52 (2024), 576–657.

- Second order fractional mean-field SDEs with singular kernels and measure initial data.
  with **Michael Röckner** and **Xicheng Zhang**.
  To appear in Ann. Probab. (2024+) arXiv: 2302.04392

- Propagation of chaos for moderately interacting particle systems related to singular kinetic Mckean-Vlasov SDEs.
  with **Jean-Francois Jabir**, **Stéphane Menozzi**, **Michael Röckner**, **Xicheng Zhang**.
  arXiv: 2405.09195

- Quantitative approximation of kinetic SDEs: From discrete to continuum.
  with **Khoa Lê** and **Chengcheng Ling**.
  In preparation.

Motivation

- N-particle systems

$$\begin{cases} dX_t^{N,i} = V_t^{N,i}\,dt \\ dV_t^{N,i} = b(t, Z_t^{N,i})\,dt + \frac{1}{N}\sum_{j\neq i} K(Z_t^{N,i} - Z_t^{N,j})\,dt + dL_t^i, \end{cases}$$

  - $Z_t^{N,i} = (X_t^{N,i}, V_t^{N,i})$: position and velocity of the ith particle at time t;
  - b (some noise): random environment;
  - $\frac{1}{N}$: mean-field scaling
  - K: interaction kernel.
  - $\{L_t^i\}_{i=1}^{\infty}$ (a family of i.i.d. α-stable processes): random phnomenent.

Motivation

- Propagation of chaos: (Mckean-Vlasov SDEs)

$$\begin{cases} dX_t = V_t\, dt \\ dV_t = b(t, Z_t)\, dt + K * \mu_t(Z_t)\, dt + dL_t, \end{cases}$$

  - μ_t is the time marginal law of the solution Z.
  - Kac 1956, ..., Sznitman 1991, ..., Jabin-Wang 2016, 2018, Lacker 2021, ...

- Second order system:

$$d\dot{X}_t = b(t, X_t, \dot{X}_t)\, dt + K * \mu_t(X_t, \dot{X}_t)\, dt + dL_t,$$

  - $\dot{X}_t = \frac{dX_t}{dt}$.

- Langevin system, kinetic system

Singular random environment

- First order system:

$$dX_t = b(X_t)\,dt + dL_t$$

  - $\alpha = 2$ (BM) (Brox, 1986) Brox diffusion:
    b is an 1-dim space white noise ($b \in C^{-\frac{1}{2}-}$);
  - $\alpha = 2$, $b \in C^{-\beta}$ with $\beta \in (\frac{1}{2}, \frac{2}{3})$
    - Delarue-Diel, 2016 rough path & 1-dim
    - Cannizzaro-Chouk, 2018 paracontrolled calculus
  - $\alpha \in (1, 2)$, $b \in C^{-\beta}$ $\beta \in ((\alpha-1)/2, (2\alpha-2)/3)$
    Kremp-Perkowski, 2022 : paracontrolled calculus (more references)

Singular kernel

- Consider the case $b \equiv 0$.
  Suppose the law μ_t has a density $f_t(x,v)$. Then by Itô's formula:

$$\partial_t f + v \cdot \nabla_x f = \Delta_v^{\frac{\alpha}{2}} f - \mathrm{div}(K * f \cdot f).$$

  - $d=3$ $K(x,v) = \nabla \frac{1}{|x|}$: Vlasov-Possion-Fokker-Planck equation
  - $d=2$ $K(x,v) = \frac{(-v_2, v_1)}{|v|^2}$: Navier-Stokes equation.
  - $d=2$, $K(x,v) = \frac{(-v_2, v_1)}{|v|^3}$: SQG systems.

Kinetic Semigroup

- Suppose $b = K \equiv 0$ and $(X_0, V_0) = (x, v)$.

$$\left(X_t^{x,v}, V_t^{x,v} \right) = \left(x + tv + \int_0^t L_s \, ds, \ v + L_t \right)$$

- Define $P_t f(x,v) := \mathbb{E} f\left(X_t^{x,v}, V_t^{x,v} \right)$

$\Rightarrow$ $u(t) := P_t \varphi + \int_0^t P_{t-s} f \, ds$ solves:

$$\partial_t u = (\Delta_v + v \cdot \nabla_x) u + f, \quad u(0) = \varphi, \quad t \geq 0.$$

- If denoting the density of $\left(\int_0^t L_s \, ds, L_t \right)$ by $p_t(x,y)$,

Then we have: $P_t f = \Gamma_t (p_t * f) = (\Gamma_t p_t) * (\Gamma_t f)$,

where $\Gamma_t f(x,v) := f(x + tv, v)$.

- **Difficulty**: i) Degenerate; ii) $\nabla_v P_t \neq P_t \nabla_v$;

Anisotriphic Besov space

- Scaling: $\left(\int_0^t L_s\,ds,\ L_t\right) \overset{d}{=} \left(t^{\frac{\alpha+1}{\alpha}} \int_0^1 L_s\,ds,\ t^{\frac{1}{\alpha}} L_1\right)$

  $\hookrightarrow$ $\boxed{x : v = (\alpha+1) : 1}$

- Anisotriphic metric:

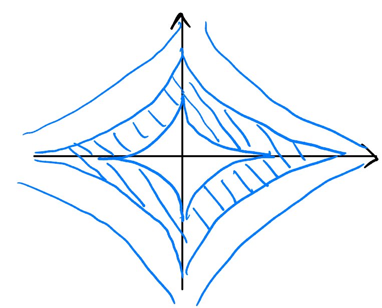
  $$|(x,v)|_a := |x|^{\frac{1}{\alpha+1}} + |v|$$

anisotriphic unit of partition:

$$\mathbb{R}^{2d} = \left(\bigcup_{j=0}^{\infty} \{ 2^{j-1} < |(\xi,\eta)|_a < 2^{j+1} \} \right) \cup \{ |(\xi,\eta)|_a < 2 \}$$

$\hookrightarrow$ constric ϕ_j^a $(j \geq -1)$

$\hookrightarrow$ $\Delta_j^a f := (\phi_j^a \hat{f})^\vee$



- Anisotrophic Besov space:

  $$\|f\|_{B^{s;a}_{\vec{p},q}} := \left(\sum_{j=-1}^{\infty} 2^{sqj} \|\Delta_j^a f\|_{\vec{p}}^q \right)^{1/q}$$

  - $s \in \mathbb{R}$, $q \in [1,\infty]$, $\vec{p} = (p_x, p_v) \in [1,\infty]^2$,
  - $$\|f\|_{\vec{p}} := \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,v)|^{p_x} dx \right)^{\frac{p_v}{p_x}} dv \right]^{1/p_v}$$

Anisotrophic Besov space

- $s > 0$,

  - $\boxed{B^{s;a}_{\vec{p},q} \simeq H^{s/1+\alpha}_{\vec{p},x} \cap H^{s}_{\vec{p},v}}$

    $$= (1-\Delta_x)^{-\frac{s}{2(1+\alpha)}} L^{\vec{p}} \cap (1-\Delta_v)^{-\frac{s}{2}} L^{\vec{p}}$$

  - $\boxed{B^{s;a}_{\infty,\infty} \simeq C^{\frac{s}{1+\alpha}}_{x} \cap C^{s}_{v}}$ $\quad C^{s}_{a} := B^{s;a}_{\infty,\infty}$

- $a \cdot \frac{d}{\vec{p}} := (1+\alpha)\frac{d}{p_x} + \frac{d}{p_v}$.

Hölder drift

- Consider the case:

  $K \equiv 0$, and b is Hölder continuous.

  - $\alpha = 2$ (BM) Chaudru de Raynal, 2015:

    $b \in C_x^{\frac{2}{3}+} \cap C_v^{0+} \Rightarrow$ strong existence + pathwise uniqueness.

  - $\alpha = 2$ Chaudru de Raynal – Honoré – Menozzi, 2018

    chain case

<u>Theorem 1</u> (CH. – Wu – Zhang, 2020)

Assume $\alpha \in (\frac{\sqrt{5}-1}{2}, 2)$ and $b \in C_x^{\frac{1+\alpha/2}{1+\alpha}+} \cap C_v^{1-\frac{\alpha}{2}+}$.

Then there is a unique strong solution.

Theorem 1 (CH.-Wu-Zhang, 2020)

Assume $\alpha \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$ and $b \in C_x^{\frac{1+\alpha/2}{1+\alpha}+} \cap C_v^{1-\frac{\alpha}{2}+}$.

Then there is a unique strong solution.

- Method: Zvonkin's transformation
- Key point: $\|P_t f\|_{C_a^{\beta+\theta}} \lesssim t^{-\frac{\theta}{\alpha}} \|f\|_{C_a^{\beta}}$.

$$\Delta_j^a P_t \approx P_t \Delta_j^a$$

$$\Delta_j^a P_t = \sum_{\ell=-1}^{\infty} \Delta_j^a P_t \Delta_\ell^a = \sum_{\ell \sim j} \Delta_j^a P_t \Delta_\ell^a .$$

Sobolev drifts

- Assume $K \equiv 0$ and b is some Sobolev function. Suppose $\alpha = 2$.
  - (Zhang 2018, Fedrizzi-Flandoli-Priola, 2017)
    $(1-\Delta_x)^{\frac{1}{3}} b \in L^p(\mathbb{R}_+, L^p(\mathbb{R}^{2d}))$, $p \in (2(2d+1), \infty) \Rightarrow$ strong well-posedness.
  - (Chaudru de Raynal - Menozzi, 2021)
    $b \in L^q(\mathbb{R}_+, L^p(\mathbb{R}^{2d}))$, $\frac{2}{q} + \frac{4d}{p} < 1 \Rightarrow$ weak well-posedness
  - (Ren-Zhang, 2024)
    b is in Kato class (one example $b \in L^q(\mathbb{R}_+, L^{\vec{p}}(\mathbb{R}^{2d}))$, $\frac{2}{q} + \frac{3d}{p_x} + \frac{d}{p_v} < 1$)
    $\Rightarrow$ weak well-posedness.

Euler approximation

- Let $n \in \mathbb{N}$ and $\pi_n(t) := \frac{[nt]}{n}$, $t \in \mathbb{R}_+$

Consider the following taming Euler scheme

$$d\dot{X}^n_t = \Gamma_{t-\pi_n(t)} b_n(t, X^n_{\pi_n(t)}, \dot{X}^n_{\pi_n(t)})dt + dW_t. \quad (Z^n_t := (X^n_t, \dot{X}^n_t))$$

$\circ$ $b_n(t,x,v) = \rho_n * b_n(t,\cdot)(x,v)$, $\rho_n(x,v) = n^{2d\vartheta}\rho(n^{\frac{3}{2}\vartheta}x, n^{\frac{\vartheta}{2}}v)$.

Theorem 2 (H.–Lê–Ling, 2024+)

i) Assume $b \in L^\infty(\mathbb{R}_+, L^{\vec{p}}(\mathbb{R}^2))$ with $a\cdot\frac{d}{\vec{p}} < 1$. $\vartheta \in (0, (2\cdot(a\cdot\frac{d}{\vec{p}}))^{-1})$

Then: $\int_0^t \| \mathbb{P}\circ(Z^n_s)^{-1} - \mathbb{P}\circ(Z_s)^{-1} \|^2_{var} ds \lesssim n^{-1} + n^{-2\vartheta}$.

ii) Assume $(1-\Delta_x)^{\frac{2}{3}} b \in L^\infty(\mathbb{R}_+, B^{\beta;a}_{\vec{p},\infty})$ with $a\cdot\frac{d}{\vec{p}} < 1$ and $\beta \in (0,1)$.

Then for $\forall \varepsilon > 0$, $\vartheta \in (0, 1/2(a\cdot\frac{d}{\vec{p}}))$,

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]} |Z^n_t - Z_t|^p\right]\right)^{\frac{1}{p}} \lesssim n^{-\frac{1+\beta/3}{2}+\varepsilon} + n^{-\vartheta(1+\beta-a\cdot\frac{d}{\vec{p}})+\varepsilon}$$

Theorem 2 (H.–Lê–Ling, 2024+)

i) Assume $b \in L^\infty(\mathbb{R}_+, L^{\vec{p}}(\mathbb{R}^{2d}))$ with $a \cdot \frac{d}{\vec{p}} < 1$. $\vartheta \in (0, (2a \cdot \frac{d}{\vec{p}})^{-1})$.

$$\int_0^t \| \mathbb{P} \circ (Z_s^n)^{-1} - \mathbb{P} \circ (Z_s)^{-1} \|_{var}^2 ds \lesssim n^{-1} + n^{-2\vartheta}.$$

ii) Assume $(1-\Delta_x)^{\frac{2}{3}} b \in L^\infty(\mathbb{R}_+, B^{\beta;a}_{\vec{p};\infty})$ with $a \cdot \frac{d}{\vec{p}} < 1$ and $\beta \in (0,1)$.

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]} |Z_t^n - Z_t|^p \right] \right)^{\frac{1}{p}} \lesssim n^{-\frac{1+\beta/3}{2}+\varepsilon} + n^{-(\vartheta+\beta)+\varepsilon}.$$

- $\begin{cases} |f(X_t) - f(X_{\pi_n(t)})| \lesssim n^{-1} \\ |f(X_t) - \Gamma_{t-\pi_n(t)} f(X_{\pi_n(t)})| \lesssim n^{-\frac{3}{2}} \end{cases}$ $\quad (X_t, V_t) = (\int_0^t W_s ds, W_t)$
- (Weak convergence) Faster than the rate $(1-\frac{d}{p})$ in (Jourdain-Menozzi, 2024).
- (Strong convergence) Extends (Lê–Ling, 2021).

$C_a^{-\frac{1}{2}-}$ drifts

- Assume $\alpha = 2$ and $K \equiv 0$.
- Let $b \in C_a^{-\beta}$, $\beta \in (\frac{1}{2}, \frac{2}{3})$
- ill-posedness: $\partial_t u = (\Delta_v + v\cdot\nabla_x)u + b\cdot\nabla_v u + f$

$$b\cdot\nabla_v u: \quad C_a^{-\beta} \times C_a^{1-\beta} \qquad (1-2\beta<0)$$

- Consider Gaussian noise b in the whole space, so that

$$b\cdot\nabla_v \int_0^t P_{t-s} b\, ds \text{ makes sense a.e.}$$

H.-Zhang-Zhu-Zhu, 2024

- Construct the paracontrolled solution to kinetic PDEs.
  (Difficulty: P_t is not a Fourier multiplier $\rightsquigarrow$ consider a new commutator)
- Establish the local method for paracontrolled solution
  (Previous paper concerning the whole space: exp weight (Hairer-Labbé)).
- Find the condition on the Gaussian noise b, with which $b \cdot \nabla_v \int_0^t P_{t-s} b \, ds$ makes sense.
  (The 0th-Wiener chaos is not 0, which is essentially different from the non-degenerate case)
- Obtain the entropy estimate for paracontrolled solution with weight.

Singular kernels

- Assume $\alpha \in (1,2]$ and $b \equiv 0$.

$$d\dot{X}_t = K * \mu_t(X_t, \dot{X}_t)dt + dL_t, \quad \mu_t = \mathbb{P} \circ (X_t, \dot{X}_t)^{-1}. \qquad (*)$$

- **(H**(K, μ_0)**)** $K \in L^q(\mathbb{R}_+, B^{\beta;a}_{\vec{p},\infty}) + \mu_0 \in B^{\beta_0;a}_{\vec{p}_0,\infty}$.

  - $\beta_0 \in (-1,0)$

$$\boxed{\frac{2}{q} - \beta + a \cdot \frac{d}{\vec{p}} - \beta_0 + a \cdot \frac{d}{\vec{p}_0} < \alpha - 1 + (\alpha+2)d.}$$

- if $\quad (\alpha+2)d + \beta_0 - a \cdot \frac{d}{\vec{p}_0} > 0$

  then $\quad \frac{2}{q} - \beta + a \cdot \frac{d}{\vec{p}} > \alpha - 1 \rightsquigarrow$ Supercritical!!!

Singular kernels

- Assume $\alpha \in (1,2]$ and $b \equiv 0$.

$$d\dot{X}_t = K * \mu_t(X_t, \dot{X}_t)dt + dL_t, \quad \mu_t = \mathbb{P} \circ (X_t, \dot{X}_t)^{-1}. \qquad (*)$$

- $(\mathbf{H}(K,\mu_0))$ $K \in L^q(\mathbb{R}_+, B^{\beta;a}_{\vec{p},\infty}) + \mu_0 \in B^{\beta_0;a}_{\vec{p}_0,\infty}$.
  - $\beta_0 \in (-1,0)$

$$\boxed{\frac{2}{q} - \beta + a\cdot\frac{d}{\vec{p}} - \beta_0 + a\cdot\frac{d}{\vec{p}_0} < \alpha - 1 + (\alpha+2)d.}$$

- Example: $\beta_0 = 0$, $\vec{p}_0 = (\infty,\infty)$, $q = \infty$, $K = K(x)$ $(p_v = \infty)$

$$\rightsquigarrow \quad -\beta + \frac{(1+\alpha)d}{p_x} < \alpha - 1 + (\alpha+2)d.$$

$K \approx \nabla^{\alpha-1+d}\delta_0$, $\quad K(x) \approx \frac{1}{|x|^\gamma}$ $\quad \gamma < \frac{\alpha-1+(\alpha+2)d}{1+\alpha}$ $(>d)$

Propagation of chaos.

- Consider the following moderately interacting particle systems:

$$\begin{cases} dX_t^{N,i} = V_t^{N,i}\,dt \\ dV_t^{N,i} = \frac{1}{N}\sum_{j\neq i}(K*\phi_t^N)(t, X_t^{N,i}-X_t^{N,j}, V_t^{N,i}-V_t^{N,j})\,dt + dL_t^i, \end{cases}$$

∘ $\phi_t^N(x,v) = N^{(2+\alpha)d\vartheta}(\Gamma_t\phi)(N^{(1+\alpha)\vartheta}x, N^{\vartheta}v)$, $\vartheta > 0$.

$\hookrightarrow \lim_{N\to\infty}\phi_t^N = \delta_0$

- Let $m_\alpha := 1/[(p_{x,0}\wedge p_{v,0}\wedge 2)\vee\alpha]$

$\theta_\alpha := [a\cdot\frac{d}{\vec{p}_0}]\vee[a\cdot\frac{d}{\vec{p}} - (1+\alpha)\beta]$

H.-Jabir-Menozzi-Röcker-Zhang. 2024

Let $\alpha \in (1,2)$

- Assume $H(K,\mu_0)$ holds with $q=\infty$, $\vec{p}<(\infty,\infty)$ and $\vec{p}_0 > (\alpha,\alpha)$.

  Suppose $\vartheta \in (0, \frac{1-m_\alpha}{\theta_\alpha})$.

  Then $\| \mathbb{P}\circ(X_t^{N,1}, V_t^{N,1})^{-1} - \mathbb{P}\circ(X_t, V_t)^{-1} \|_{var} \lesssim N^{-r}$,

  where $r = r(\alpha, \beta, \vec{p}, \beta_0, \vec{p}_0, K, \mu_0) > 0$.

- Moreover, if $H((1-\Delta_x)^{\frac{\alpha}{2(1+\alpha)}} K, \mu_0)$ holds.

  Then $\| \sup_{t\in[0,T]} |Z_t^{N,1} - Z_t^1| \|_{L^2(\Omega)} \lesssim N^{-r}$.

- The proof is based on the technique used in Olivera-Richard-Tomašević, 2021 and H.-Röcker-Zhang 2024+.